Improving the Convergence of NN Effective Field Theory

Daniel R. Phillips[a], Gautam Rupak[a] and Martin J. Savage[a,b]

[a] *Department of Physics, University of Washington,*
Seattle, WA 98195.

[b] *Jefferson Lab., 12000 Jefferson Avenue, Newport News,*
Virginia 23606.

Abstract

We study a low-energy effective field theory (EFT) describing the NN system in which all exchanged particles are integrated out. We show that fitting the residue of the $^3S_1$ amplitude at the deuteron pole, rather than the $^3S_1$ effective range, dramatically improves the convergence of deuteron observables in this theory. Reproducing the residue ensures that the tail of the deuteron wave function, which is directly related to NN scattering data via analytic continuation, is correctly reproduced in the EFT at next-to-leading order. The role of multi-nucleon-electroweak operators which produce deviations from effective-range theory can then be explicitly separated from the physics of the wave function tail. Such an operator contributes to the deuteron quadrupole moment, μ_Q, at low order, indicating a sensitivity to short-distance physics. This is consistent with the failure of impulse-approximation calculations in NN potential models to reproduce μ_Q. The convergence of NN phase shifts in the EFT is unimpaired by the use of this new expansion.

arXiv:nucl-th/9908054v1 17 Aug 1999

When the deuteron is probed at low momentum-transfers it is appropriate to treat the nuclear bound state in a long-wavelength approximation. In such an approximation no details of the short-distance dynamics are observed. Consequently most observables are dominated by the physics of the deuteron's wave function tail. Ignoring, for the time being, the deuteron's D-state, two numbers characterize this tail. One is the deuteron binding energy, which determines the rate of exponential fall-off of the wave function. The other is the asymptotic S-state normalization, A_S, which multiplies this exponential. In the limit as $r \to \infty$ the wave function becomes

$$\psi(\mathbf{r}) \quad \to \quad \frac{A_S}{\sqrt{4\pi}} \frac{e^{-\gamma r}}{r} \equiv \sqrt{\frac{\gamma Z_d}{2\pi}} \frac{e^{-\gamma r}}{r} \quad . \tag{1}$$

Below we work in terms of the dimensionless quantity Z_d, rather than $A_S \equiv \sqrt{2\gamma Z_d}$. Neither Z_d nor A_S can be measured directly in experiment, but their values can be determined by analytic continuation of the NN scattering amplitude into the bound-state region. Parameterizing the $^3S_1$ phase shifts as [1]

$$|\mathbf{k}| \cot \delta_0 = -\gamma + \frac{1}{2}\rho_d(|\mathbf{k}|^2 + \gamma^2) + w_2 (|\mathbf{k}|^2 + \gamma^2)^2 + \dots \quad , \tag{2}$$

with $\gamma^{-1} = 4.319$ fm, $\rho_d = 1.765(4)$ fm, and $w_2 = 0.389$ fm^3 [2,3], reproduces the phase-shift data in the low-momentum region. Near the deuteron pole, the scattering amplitude becomes

$$\lim_{k \to i\gamma} \frac{4\pi}{M_N} \frac{1}{|\mathbf{k}| \cot \delta_0 - i|\mathbf{k}|} = -\frac{4\pi}{M_N} \frac{1}{1 - \gamma\rho_d} \frac{1}{ik + \gamma} + R(k) \equiv -\frac{4\pi}{M_N} \frac{Z_d}{ik + \gamma} + R(k) \quad . \tag{3}$$

Here the quantity $R(k)$ is regular in the limit $k \to i\gamma$ and, importantly, is the only place that higher-order terms in the expansion of $|\mathbf{k}| \cot \delta_0$ enter this amplitude. It follows that the residue Z_d is given by

$$Z_d = \frac{1}{1 - \gamma\rho_d} \quad . \tag{4}$$

The measured values of γ and ρ_d quoted above give $Z_d = 1.690(3)$, which is also the value of Z_d obtained using A_S given in the 1993 Nijmegen phase-shift analysis [2,3], as required.

In the 50 years since effective range theory was first formulated many calculations have taken the "effective-range-theory wave function" given in eq. (1) as the wave function for *all* r and calculated low-momentum processes involving the deuteron. As we will recapitulate below, the results obtained by this procedure are often in good agreement with both experimental data and modern NN potential models. This suggests that such quantities truly are insensitive to details of the short-distance deuteron dynamics.

The central tenet of effective field theory (EFT) is that low-momentum probes of any system should not reveal details of that system's short-distance dynamics. Much effort has recently been invested in calculating observables in the two-nucleon system using effective field theory techniques based on this tenet [4]– [43]. An extreme, but efficacious, EFT has been constructed for the NN system by considering only distance scales $r \ll 1/m_\pi$, i.e. the region where the S-state deuteron wave function is given by eq. (1). In this EFT the

low-momentum scales in the problem are γ and the typical nucleon momentum k. It is a non-relativistic theory of nucleons interacting via contact interactions, which we denote by EFT($\pi\!\!\!/$). These contact interactions represent the results of integrating the pions, and all other exchanged degrees of freedom, out of the theory. The leading contact interaction is responsible for binding the deuteron, and it is enhanced over expectations based on naive dimensional analysis, as are all other S-wave contact interactions. Relativistic corrections can be included perturbatively, and are very small. We shall not discuss them here, the interested reader should consult Ref. [36] for details. The Lagrangian describing the strong interactions in EFT($\pi\!\!\!/$) is

$$\mathcal{L} = -\pi\!\!\!/C_0^{(^3S_1)}\left(N^T P_i N\right)^\dagger \left(N^T P_i N\right) - \pi\!\!\!/C_2^{(^3S_1)}\frac{1}{2}\left[(N^T P_i N)^\dagger \left(N^T \mathcal{O}_2^{jj,i} N\right) + h.c.\right] + \ldots \quad, \quad (5)$$

where P_i is the spin-isospin projector for the $^3S_1$ channel [13,15] and the ellipses denote terms involving more spatial derivatives. The two derivative operator, $\mathcal{O}_2^{xy,j}$, is given by

$$\mathcal{O}_2^{xy,j} = -\frac{1}{4}\left(\overleftarrow{\mathbf{D}}^x\overleftarrow{\mathbf{D}}^y P^j + P^j\overrightarrow{\mathbf{D}}^x\overrightarrow{\mathbf{D}}^y - \overleftarrow{\mathbf{D}}^x P^j\overrightarrow{\mathbf{D}}^y - \overleftarrow{\mathbf{D}}^y P^j\overrightarrow{\mathbf{D}}^x\right) \quad. \quad (6)$$

The coefficients C_0, C_2, ... are to be fit to data. Until now this has been done by reproducing the effective-range expansion (2) up to some given order in the expansion parameter $Q \sim k, \gamma$. This is straightforward, since the exact result for $|\mathbf{k}|\cot\delta$ in EFT($\pi\!\!\!/$), if relativity and S-D mixing are ignored, is

$$-|\mathbf{k}|\cot\delta_0 = \frac{4\pi}{M_N}\frac{1}{\sum_n C_{2n}|\mathbf{k}|^{2n}} + \mu \quad, \quad (7)$$

when dimensional regularization with power-law divergence subtraction is employed [13]. Expanding each C_{2n} in powers of Q itself, i.e. writing: $C_{2n} = C_{2n,-n-1} + C_{2n,-n} + \ldots$, and then expanding eq. (7) in powers of Q one can match to eq. (2) order-by-order in Q. If ρ_d and w_2 are both taken to scale as Q^0, then at next-to-next-to-leading order (N^{2}LO) in Q this yields the coefficients

$$\pi\!\!\!/C_{0,-1}^{(^3S_1)} = -\frac{4\pi}{M_N}\frac{1}{(\mu-\gamma)} \quad, \quad \pi\!\!\!/C_{0,0}^{(^3S_1)} = \frac{2\pi}{M_N}\frac{\gamma^2\rho_d}{(\mu-\gamma)^2} \quad, \quad \pi\!\!\!/C_{0,1}^{(^3S_1)} = -\frac{\pi}{M_N}\frac{\gamma^4\rho_d^2}{(\mu-\gamma)^3} \quad,$$

$$\pi\!\!\!/C_{2,-2}^{(^3S_1)} = \frac{2\pi}{M_N}\frac{\rho_d}{(\mu-\gamma)^2} \quad, \quad \pi\!\!\!/C_{2,-1}^{(^3S_1)} = -\frac{2\pi}{M_N}\frac{\gamma^2\rho_d^2}{(\mu-\gamma)^3} \quad, \quad \pi\!\!\!/C_{4,-3}^{(^3S_1)} = -\frac{\pi}{M_N}\frac{\rho_d^2}{(\mu-\gamma)^3} \quad, \quad (8)$$

at the renormalization scale μ. The dominant interaction $\pi\!\!\!/C_{0,-1}^{(^3S_1)}$ must be iterated to all orders to generate the leading order (LO) amplitude. All other interactions, including the corrections to C_0, can be treated in perturbation theory. At next-to-leading order (NLO) and above the coefficients are chosen to reproduce ρ_d exactly. Hence we refer to this choice as the "ρ-parameterization. With this choice of C's, observables are power-series expansions in the quantity $\gamma\rho_d$ order-by-order in the Q expansion. In particular, the expansion of Z_d is

$$Z_d^{(Q)} = 1 + \gamma\rho_d + (\gamma\rho_d)^2 + (\gamma\rho_d)^3 + \ldots \quad. \quad (9)$$

Physically, the amplitude for any low-energy elastic reaction on the deuteron will always include an overall factor of Z_d. Thus the appearance of this series in a number of previous

calculations in EFT($\not\pi$) should come as no surprise. Indeed, Park *et al.* have had considerable success in reproducing experimental data by pursuing a strategy of fitting Z_d at NLO [7,27]. In fact, such an approach is implicit in other work on EFT in nuclear physics, e.g. Refs. [5,10,29]. Recently, Phillips and Cohen [41] and Rho [43] have stressed that it is better to fit Z_d in the EFT than ρ_d, since this ensures that the calculation correctly reproduces the long-distance piece of the deuteron wave function. The difference between fitting ρ_d and Z_d is higher-order in any given EFT calculation, but, as we shall show here, in EFT($\not\pi$) fitting Z_d markedly improves the convergence of calculations for a variety of low-energy elastic processes on deuteron targets. We will also discuss inelastic processes such as $np \to d\gamma$, and show that demanding that Z_d is correctly reproduced also improves the convergence of these calculations. Finally, we will discuss the NN phase shifts in the Q-expansion when this alternative fitting procedure is employed. We will show that, despite abandoning the exact fit of ρ_d in the Q-expansion of $k \cot \delta$, the results found for the phase shift with this modified fitting procedure are no worse than those found using the "ρ-parameterization".

In this work we require that the position and residue of the deuteron pole are reproduced exactly at NLO in the Q expansion. The coefficients of the Lagrangian (5) determined by this constraint are (again neglecting relativistic effects):

$$
{}_z^{\not\pi}C_{0,-1}^{(^3S_1)} = -\frac{4\pi}{M_N}\frac{1}{(\mu-\gamma)} \quad , \quad {}_z^{\not\pi}C_{0,0}^{(^3S_1)} = \frac{2\pi}{M_N}\frac{\gamma(Z_d-1)}{(\mu-\gamma)^2} \quad , \quad {}_z^{\not\pi}C_{0,1}^{(^3S_1)} = -\frac{\pi}{M_N}\frac{(Z_d-1)^2(2\mu-\gamma)\gamma}{(\mu-\gamma)^3} \quad ,
$$

$$
{}_z^{\not\pi}C_{2,-2}^{(^3S_1)} = \frac{2\pi}{M_N}\frac{Z_d-1}{\gamma(\mu-\gamma)^2} \quad , \quad {}_z^{\not\pi}C_{2,-1}^{(^3S_1)} = -\frac{2\pi}{M_N}\frac{(Z_d-1)^2\mu}{\gamma(\mu-\gamma)^3} \quad , \quad {}_z^{\not\pi}C_{4,-3}^{(^3S_1)} = -\frac{\pi}{M_N}\frac{(Z_d-1)^2}{\gamma^2(\mu-\gamma)^3} \quad . \quad (10)
$$

We call this the "z-parameterization"[1]. Here the quantity $Z_d - 1$ is taken to be of order Q. This choice for the C's is formally equivalent order-by-order to that of eq. (8), the difference between the two always being of higher order in the Q-expansion. The advantage of this choice can be seen in the Q-expansion of Z_d. Instead of the convergence displayed in eq. (9) we now have, by explicit construction:

$$
Z_d^{(Q)} = 1 + (Z_d - 1) + 0 + 0 + 0 + \dots \quad . \tag{11}
$$

Both parameterizations yield $Z_d^{(LO)} = 1$. However, at NLO the two series in eqs. (9) and (11) have very different behavior. In the ρ-parameterization the first $\gamma\rho_d$ correction brings that expansion perturbatively close to the complete sum of Z_d, with corrections of order $(\gamma\rho_d)^2$ and higher. In contrast, in eq. (11) the first $(Z_d - 1)$ correction brings the series into exact agreement with the complete sum, and all higher-order terms vanish. This behavior translates directly into the convergence of the Q-expansion for observables involving the deuteron.

The easiest place to see this is in the deuteron charge form factor F_C. In the ρ-parameterization, neglecting relativistic effects gives LO, NLO, and N^{2}LO contributions to the charge form factor which are [36],

---

[1]Pronounced "zed"-parameterization.

4

$$F_C^{\rho(0)}(|\mathbf{q}|) = \frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right) \quad , \quad F_C^{\rho(1)}(|\mathbf{q}|) = -\gamma\rho_d\left[1 - \frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right)\right] \quad ,$$

$$F_C^{\rho(2)}(|\mathbf{q}|) = -\gamma^2\rho_d^2\left[1 - \frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right)\right] - \frac{1}{6}r_N^2\,|\mathbf{q}|^2\,\frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right) \quad , \qquad (12)$$

where $|\mathbf{q}|$ is the momentum transfer, and $r_N = 0.790\pm0.012$ fm is the isoscalar charge radius of the nucleon. In contrast, the z-parameterization yields LO, NLO and N^{2}LO contributions to the form factor:

$$F_C^{z(0)}(|\mathbf{q}|) = \frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right) \quad , \quad F_C^{z(1)}(|\mathbf{q}|) = -(Z_d - 1)\left(1 - \frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right)\right) \quad ,$$

$$F_C^{z(2)}(|\mathbf{q}|) = -\frac{1}{6}r_N^2\,|\mathbf{q}|^2\,\frac{4\gamma}{|\mathbf{q}|}\tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right) \quad . \qquad (13)$$

Comparing the expressions for F_C in the ρ and z parameterizations it is clear that demanding that Z_d be reproduced exactly at NLO means that the *only* correction at N^{2}LO is due to the finite size of the nucleon. There are no corrections resulting from "wave function" effects at any order beyond NLO.

This can be clearly understood if we assume the wave function given in eq. (1), is correct for all r, calculate F_C via:

$$F_C(|\mathbf{q}|) = \int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}/2}|\psi(\mathbf{r})|^2, \qquad (14)$$

and also add an additional (constant) piece to F_C which ensures that $F_C(0) = 1$. When this is done we reproduce eq. (13), but with $r_N = 0$. In other words, at NLO the z-parameterization exactly reproduces the results obtained in effective range theory calculations of a deuteron containing point-like nucleons. However, EFT also allows for systematic incorporation of the effects of finite-nucleon size, via terms such as those proportional to r_N^2 above, in a way that maintains the locality of the field theory.

Furthermore, one order beyond that calculated here, i.e. at N^{3}LO, a four-nucleon-one-photon interaction contributes to F_C and shifts the deuteron charge radius from its impulse approximation value. This contribution to F_C is beyond the scope of effective range theory. In contrast, the charge radius extracted from eq. (13) at N^{2}LO is

$$\langle r_d^2\rangle^{\mathrm{EFT}} = r_N^2 + \frac{Z_d}{8\gamma^2} = 4.566\text{ fm}^2 \quad . \qquad (15)$$

We estimate the error from the next-order corrections (including the counterterm) to be $\sim 3\%$. In Table I we compare this, and other static properties of the deuteron, to those obtained in the potential models Nijm93, Reid93 [44], and OBEPQ [45] (energy-independent version), in impulse approximation, and also to the experimental value. We see that in the case of $\langle r_d^2\rangle$ the EFT calculation agrees with data and potential model calculations within 1–2 %. Note that in comparing results we have assumed a common contribution of $r_N^2 = (0.79\text{ fm})^2$ to $\langle r_d^2\rangle$.

In Table I we have also displayed results for the deuteron's quadrupole moment, μ_Q, and for the quadrupole "radius", $\langle r_Q^2\rangle$. We now spend some time discussing the results obtained

Calculation	$\langle r_d^2 \rangle$ (fm^2)	μ_Q (fm^2)	$\langle r_Q^2 \rangle$ (fm^4)
EFT (LO)	2.332	0.335	0.70
EFT (NLO)	4.566	0.286 (fit)	1.40
OBEPQ	4.499	0.274	1.27
Nijm93	4.489	0.271	1.22
Reid93	4.501	0.270	1.22
Experiment	4.493(23)	0.2859(3)	

TABLE I. Results for the square of the deuteron charge radius, $\langle r_d^2 \rangle$, the deuteron quadrupole moment, μ_Q, and the square of the quadrupole "radius", $\langle r_Q^2 \rangle$, at LO and NLO, compared with impulse-approximation potential-model calculations, and the experimental results. The experimental value of $\langle r_d^2 \rangle$ is taken from Ref. [46]. See also the more recent Refs. [47,48].

in EFT($\not\pi$) for the deuteron's quadrupole form factor. This quantity clearly demonstrates the limitations of effective range theory.

Naively one might think that it is impossible for the deuteron to acquire a quadrupole moment in this EFT, but such is not the case. The deuteron becomes non-spherical due to the appearance in the theory of contact operators with two derivatives which mix the S and D-states. In Ref. [38] it was shown that the coefficient of this operator could be fit to the asymptotic D-to-S ratio, η_{sd}. The value of η_{sd} is known to high precision from analytic continuation of the S-D mixing parameter, $\overline{\varepsilon}_1$: $\eta_{sd} = 0.02544 \pm 0.0002$ [3]. Fitting η_{sd} means that the tail of the D-wave piece of the deuteron wave function will be correct. The leading order result for F_Q is then just the same as that obtained assuming asymptotic forms for the deuteron radial wave functions, working to leading order in η_{sd}, and setting $Z_d = 1$

$$\frac{1}{M_d^2} F_Q^{z(0)}(|\mathbf{q}|) = -\frac{3\eta_{sd}}{2\sqrt{2}\gamma|\mathbf{q}|^3} \left[4|\mathbf{q}|\gamma - \left(3|\mathbf{q}|^2 + 16\gamma^2\right) \tan^{-1}\left(\frac{|\mathbf{q}|}{4\gamma}\right) \right] . \qquad (16)$$

At this order the deuteron quadrupole moment is $\mu_Q^{(LO)} = \eta_{sd}/(\sqrt{2}\gamma^2) = 0.335$ fm^2, determined from the $|\mathbf{q}| \to 0$ limit of eq. (16) (see also Ref. [49]).

In the z-parameterization at NLO for F_Q corrections proportional to $Z_d - 1$ appear. Also at NLO there is a contribution from a four-nucleon-one-quadrupole-photon local operator whose coefficient is not related by gauge invariance to the NN scattering phase shift data [36]. This coefficient must be determined from the deuteron quadrupole moment. An overall shift in the value of F_Q, which we denote by $\delta\mu_Q$, results from the inclusion of this operator. The total NLO contribution to F_Q is

$$\frac{1}{M_d^2} F_Q^{z(1)}(|\mathbf{q}|) = \delta\mu_Q + (Z_d - 1) \left[\frac{1}{M_d^2} F_Q^{z(0)}(|\mathbf{q}|) - \mu_Q^{(LO)} \right] . \qquad (17)$$

The numerical value of $\delta\mu_Q$ required to reproduce the measured value of the deuteron quadrupole moment is $\delta\mu_Q = -0.0492$ fm^2. The EFT analyses of Refs. [36,38] suggest that a counterterm of this size is expected from renormalization group arguments and naturalness. As it is the contribution of a two-body quadrupole charge operator to F_Q it simply
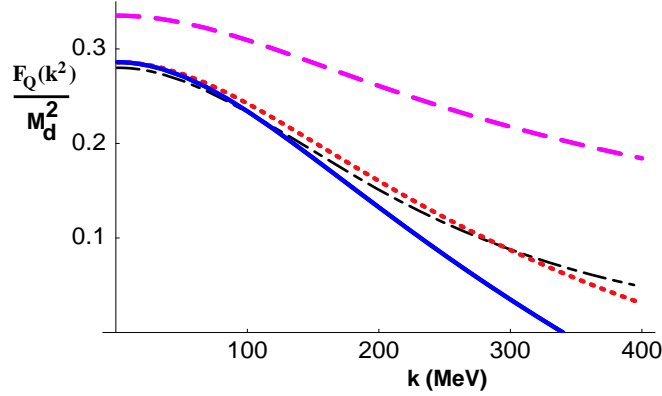
6

FIG. 1. *The deuteron quadrupole form factor. The dashed, dotted and solid curves correspond to the LO, NLO and* N^2LO *contributions in the z-parameterization. The dot-dashed curve corresponds to a calculation with the Bonn-B potential [52] in the formulation of [53] .*

does not appear in effective range theory. In Refs. [36,41] it was argued that the absence of such a term could be responsible for "modern" potential models' persistent underprediction of μ_Q by ~ 5 % in impulse approximation [44,50,51]. The preeminent role played by this counterterm, as compared to others in the NN effective field theory, is because μ_Q is more sensitive to short-distance physics than many other deuteron observables.

As in the charge form factor, the only correction to this result at N²LO comes from the finite-size of the nucleon.

$$\frac{1}{M_d^2} F_Q^{z(2)}(|\mathbf{q}|) = -\frac{1}{6} r_N^2 |\mathbf{q}|^2 \frac{1}{M_d^2} F_Q^{z(0)}(|\mathbf{q}|) \quad , \tag{18}$$

All the nuclear effects have already been taken care of by the demand that Z_d be replicated exactly.

The form factor computed at LO, NLO and N²LO is shown in fig. 1, together with a potential-model calculation which used the Bonn-B potential [52] in the formulation of [53]. The difference between $F_Q(|\mathbf{q}| = 0)$ for the two higher-order EFT($\not\pi$) calculations and the potential model calculation shown in fig. 1, occurs because the calculation using the Bonn-B potential does not produce the correct deuteron quadrupole moment.

We can also easily compute the deuteron quadrupole radius, defined by

$$\langle r_Q^2 \rangle \equiv -6 \frac{dF_Q}{dq^2} \bigg|_{q^2=0} \quad . \tag{19}$$

At N²LO there is no counterterm contribution to $\langle r_Q^2 \rangle$, and so we expect that it will be less sensitive to short-distance physics than μ_Q. Specifically:

$$\langle r_Q^2 \rangle = \frac{9 \, \eta_{sd}}{80\sqrt{2}\gamma^4} \left(Z_d + \frac{80}{9} r_N^2 \gamma^2 \right) = 1.40 \text{ fm}^4 \quad . \tag{20}$$

At next order a contribution proportional to η_{sd}^2 enters, and a two-body counterterm also appears. The error in our result for $\langle r_Q^2 \rangle$ is dominated by the counterterm, which produces

a relative error $(\gamma/m_\pi)^5 1/\eta_{sd} \sim 15\ \%$. This is roughly the discrepancy between eq. (20) and the results obtained from potential models with similar values of η_{sd}.

Similar results to those shown here for F_C and F_Q can also be obtained for the deuteron magnetic form factor, F_M. It too has good convergence properties in the z-parameterization of EFT($\not{\pi}$).

We now leave the form factors for elastic electron-deuteron scattering and instead turn our attention to a quantity which is very much dominated by the long-distance physics of the deuteron. The electric polarizability of the deuteron has been computed in EFT($\not{\pi}$) up to N^{3}LO in the ρ-parameterization [36,38,42]:

$$
\begin{aligned}
\alpha_{E0}^{(\rho)} &= \frac{\alpha M_N}{32\gamma^4}\left[1 \ + \ \gamma\rho_d \ + \ \gamma^2\rho_d^2 \ + \ \gamma^3\rho_d^3 \ + \ \frac{2\gamma^2}{3M_N^2} \ + \ \frac{M_N\gamma^3}{3\pi}D_P\right] \\
&= \quad 0.377 \ + \ 0.153 \ + \ 0.062 \ + \ 0.022 \ + \ 0.0006 \ - \ 0.0036 \\
&= 0.616\ \mathrm{fm}^3 \quad ,
\end{aligned}
\tag{21}
$$

where $D_P = \not{C}_2^{(^3P_0)} + 2\ \not{C}_2^{(^3P_1)} + \frac{20}{3}\ \not{C}_2^{(^3P_2)} \ = \ -1.51\ \mathrm{fm}^3$ is the combination of P-wave coefficients that contributes to the E1 amplitude in $np \to d\gamma$. The dominant corrections here always arise from the perturbative expansion of Z_d. The relativistic correction, formally N^{2}LO in the momentum expansion, gives a contribution which numerically is of much higher order. From now on we will ignore it. The uncertainty in this calculation in the ρ-parameterization is then set by the omission of $\gamma^4\rho_d^4$ and higher. This constitutes an uncertainty of $\sim 2.5\%$ due to the expansion of Z_d, and leads to $\Delta\alpha_{E0}^{(\rho)} \sim \pm 0.015\ \mathrm{fm}^3$.

In contrast, the z-parameterization gives

$$
\begin{aligned}
\alpha_{E0}^{(z)} &= \frac{\alpha M_N}{32\gamma^4}\left[1 \ + \ (Z_d - 1) \ + \ 0 \ + \ \frac{M_N\gamma^3}{3\pi}D_P\right] \\
&= \quad 0.3770 \ + \ 0.2605 \ + \ 0 \ - \ 0.0036 \ = \ 0.6339\ \mathrm{fm}^3 \quad ,
\end{aligned}
\tag{22}
$$

rapid convergence indeed. The insertion of the P-wave operators also picks up factors of Z_d when computed to higher orders and therefore we provocatively write

$$
\begin{aligned}
\alpha_{E0}^{(z)} &= \frac{\alpha M_N}{32\gamma^4}\ Z_d\ \left[1 \ + \ \frac{M_N\gamma^3}{3\pi}D_P \ + \ ...\right] \\
&= 0.6314\ \mathrm{fm}^3 \quad ,
\end{aligned}
\tag{23}
$$

where the ellipses denote higher-order terms, which include a N^{4}LO contribution from $S-D$ mixing proportional to η_{sd}^2. The first contribution from photon-four-nucleon vertices which are not constrained by NN phase shifts or gauge invariance also occurs at N^{4}LO. We expect that these two effects combined may shift the value of α_{E0} by $\pm 0.0015\ \mathrm{fm}^3$.

This result is commensurate with the good agreement between "modern" potential model calculations which give $\alpha_{E0} = 0.6328 \pm 0.0017\ \mathrm{fm}^3$ [54,55]. The deuteron electric polarizability is very precisely predicted once the NN phase shifts are properly described. EFT thus provides a very accurate description of this quantity, as long as one demands that the tail of the deuteron wave function be well-reproduced at low order in the EFT expansion. This will be generally true for any low-energy elastic process on the deuteron. Examples of such

processes already computed in the ρ-parameterization include Compton scattering [16,17], and the reaction $\nu d \to \nu d$ [32].

We now turn our attention to reactions where the deuteron is only present in the final or initial state. In particular, we wish to see if the z-parameterization improves the convergence of these reactions, in which only one of the asymptotic states involves a deuteron.

The radiative capture process, $np \to d\gamma$, has been studied in great detail in both EFT($\not{\pi}$) [36,38,42] and the theory with pions [20]. For incident cold or thermal neutrons, the rate for this process is dominated by M1 capture from the $^1S_0$ channel via the isovector magnetic moment of the nucleon. The amplitudes for capture from other channels are strongly suppressed in this kinematic regime. Forthcoming experimental data from Grenoble has stimulated interest in the calculation of the isoscalar M1 and E2 matrix elements in $np \to d\gamma$ using different formulations of effective field theory [35,38]. In Ref. [38] these amplitudes were computed in the ρ-parameterization of EFT($\not{\pi}$). We now wish to examine them in the z-parameterization. The isoscalar M1 amplitude has been computed up to NLO, and at that order it is the same in both the ρ and z parameterizations, due to the vanishing contribution of the one-body operator. However, the isoscalar E2 amplitude is affected by the choice of parameterization. Up to N^{2}LO the amplitude $\tilde{X}_{E2_s}$, as defined in [38], becomes

$$\tilde{X}_{E2_s} = \frac{\delta\mu_Q\,\gamma^2}{4\sqrt{2}} - \frac{\eta_{sd}}{10}\left(1 + \frac{3}{8}(Z_d - 1) - \frac{7}{32}(Z_d - 1)^2\right) + \frac{3}{40}E_1^{(4)}\,\gamma^4 \quad, \tag{24}$$

when the z-parameterization is used. Here $E_1^{(4)}$ is found by fitting the shape of the S-D mixing parameter $\overline{\varepsilon}_1$ and turns out to be $E_1^{(4)} = -2.880$ fm^4 [38].

Numerically, this leads to a series for the ratio of the isoscalar E2 matrix element to the experimental value of the dominant isovector M1 matrix element:

$$\frac{\tilde{X}_{E2_S}}{\tilde{X}_{M1_V}} = -[\,1.565 \;+\; 0.693 \;+\; 0.209\,] \times 10^{-4} \;=\; -2.47 \times 10^{-4} \quad. \tag{25}$$

While the central value is essentially unchanged from that found in Ref. [38] we now estimate the uncertainty in this ratio to be $\sim 6\%$, much smaller than the $\sim 15\%$ quoted there. The series in eq. (25) appears to be converging faster than the analogous expression obtained using the ρ-parameterization [38]. The reasons for this are not entirely clear, since the naive expansion parameter, $Z_d - 1 = 0.690$, is actually *larger* than $\gamma\rho_d = 0.408$. However, it seems that, in general, an expansion of $\sqrt{1 + Z_d - 1}$ in powers of $Z_d - 1$ converges faster than one of $(1 - \gamma\rho_d)^{-1/2}$ in powers of $\gamma\rho_d$. It will be interesting to see how far the z-parameterization improves the convergence of EFT calculations of the astrophysically-important reaction $pp \to de^+\nu_e$. Kong and Ravndal have pointed out that the main obstacle to agreement between EFT and potential model calculations of this reaction is the EFT's failure to exactly reproduce the overall factor of $\sqrt{Z_d}$ which appears in the astrophysical S-factor [28].

Finally, we turn our attention to NN scattering. In this case we merely wish to reassure ourselves that the z-parameterization does no *worse* than the ρ-parameterization. We examine the $^3S_1$ phase shift $\overline{\delta}_0$ up to N^{2}LO. Taking the expression for the coefficients (10) and computing the amplitude, and hence the phase shift, up to a given order in Q, we find, successively:

$$\overline{\delta}_0 = \pi - \tan^{-1}\left(\frac{\mathbf{k}}{\gamma}\right) \;-\; (Z_d - 1)\frac{\mathbf{k}}{2\gamma} \;+\; (Z_d - 1)^2\frac{\mathbf{k}}{4\gamma} \quad, \tag{26}$$
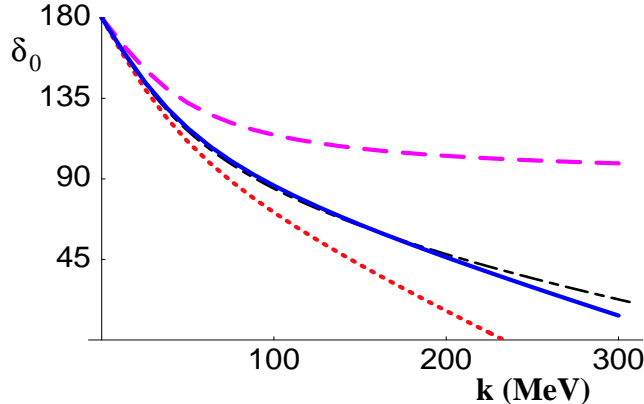
FIG. 2. *The $^3S_1$ phase shift as a function of nucleon momentum $|\mathbf{k}|$. The dashed, dotted and solid curves correspond to the LO, NLO and N^2LO contributions in the z-parameterization. The dot-dashed curve corresponds to the Nijmegen partial wave analysis [2].*

where the first two pieces are order Q^0, the second is order Q, and the third order Q^2. Relativistic corrections are again omitted. This phase shift is plotted in fig. 2. At N^2LO it agrees very well with the Nijmegen partial wave analysis [2]. The agreement is certainly as good as that obtained in the ρ-expansion [36]. It is very encouraging that even for momenta which are large in the context of this EFT, i.e. $k \sim m_\pi$, the convergence of the Q-expansion for NN scattering is good if the z-parameterization is used. Given this result, and that observables involving the deuteron appear to converge significantly faster, it is clear that the z-parameterization of the coefficients in the Lagrange density (5) is better than the ρ-parameterization that has been used up until now. Its use may also improve the convergence of three-nucleon observables such as the quartet phase shifts [33,34].

EFT($\not{\pi}$) is the most general theory in the two-nucleon sector consistent with the symmetries of the strong and electroweak interactions in the very low-energy regime. As we explained above, the only change implemented here, vis á vis previous work on this theory, is to ensure that the tail of the deuteron wave function, in both the S and D-states, is correctly reproduced at low order in the EFT when elastic processes are calculated. That this improves the convergence of the EFT is not surprising. By choosing slightly different constraints to determine the coefficients in the Lagrange density we have been able to write observables in terms of the normalization constants Z_d and η_{sd}, rigorously defined in terms of the residue of the scattering amplitude at the deuteron pole. If the coefficients are fit in this way then the full impulse approximation result in effective range theory is obtained for elastic processes at NLO in the EFT—at least up to leading order in η_{sd}. NLO EFT($\not{\pi}$) calculations of elastic processes get the long-distance piece of the nuclear dynamics correct. However, EFT($\not{\pi}$) calculations then go beyond effective range theory, in that they systematically include two-body currents which account for the short-distance physics not included in the asymptotic wave functions. The advantage of using the EFT is that these contributions, which are not constrained by NN phase shift data, have sizes set by the underlying short-distance physics. The best example of this is the four-nucleon-one-quadrupole-photon operator which contributes at NLO to the quadrupole moment.

REFERENCES

[1] H. A. Bethe, *Phys. Rev.* **76**, 38 (1949); H. A. Bethe and C. Longmire, *Phys. Rev.* **77**, 647 (1950).

[2] V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, *Phys. Rev.* C **48**, 1993 (792),

[3] J. J. de Swart, C. P. F. Terheggen, and V. G. J. Stoks, Talk presented at Third International Symposium on the Deuteron, Dubna, Russia, July 1995, `nucl-th/9509032`.

[4] S. Weinberg, *Phys. Lett.* B **251**, 288 (1990); *Nucl. Phys.* B **363**, 3 (1991); *Phys. Lett.* B **295**, 114 (1992).

[5] C. Ordonez and U. van Kolck, *Phys. Lett.* B **291**, 459 (1992); C. Ordonez, L. Ray and U. van Kolck, *Phys. Rev. Lett.* **72**, 1982 (1994); *Phys. Rev.* C **53**, 2086 (1996); U. van Kolck, *Phys. Rev.* C **49**, 2932 (1994).

[6] J. L. Friar, `nucl-th/9601012`; `nucl-th/9601013`; *Few Body Syst.* **22**, 161 (1997); `nucl-th/9804010`; J.L. Friar, D. Huber, and U. van Kolck, *Phys. Rev.* C **59**, 53 (1999).

[7] T.-S. Park, D.-P. Min and M. Rho, *Phys. Rev. Lett.* **74**, 4153 (1995) ; *Nucl. Phys.* A **596**, 515 (1996).

[8] T.D. Cohen, *Phys. Rev.* C **55**, 67 (1997). D.R. Phillips and T.D. Cohen, *Phys. Lett.* B **390**, 7 (1997). K.A. Scaldeferri, D.R. Phillips, C.W. Kao and T.D. Cohen, *Phys. Rev.* C **56**, 679 (1997); S.R. Beane, T.D. Cohen and D.R. Phillips, *Nucl. Phys.* A6324451998; D.R. Phillips, S.R. Beane and T.D. Cohen, *Annals Phys.* **263**, 255 (1998).

[9] M.J. Savage, *Phys. Rev.* C **55**, 2185 (1997).

[10] G.P. Lepage, `nucl-th/9706029`, Lectures at 9th Jorge Andre Swieca Summer School: Particles and Fields, Sao Paulo, Brazil, Feb 1997.

[11] M. Luke and A.V. Manohar, *Phys. Rev.* D **55**, 4129 (1997).

[12] D.B. Kaplan, M.J. Savage, and M.B. Wise, *Nucl. Phys.* B **478**, 629 (1996).

[13] D.B. Kaplan, M.J. Savage and M.B. Wise, *Phys. Lett.* B **424**, 390 (1998); *Nucl. Phys.* B **534**, 329 (1998).

[14] U. van Kolck, *Nucl. Phys.* A **645**, 273 (1999)

[15] D.B. Kaplan, M.J. Savage and M.B. Wise, *Phys. Rev.* C **59**, 617 (1999).

[16] J.W. Chen, H. W. Griesshammer, M. J. Savage and R. P. Springer, *Nucl. Phys.* A **644**, 221 (1999); *Nucl. Phys.* A **644**, 245 (1999).

[17] J.W. Chen, `nucl-th/9810021`.

[18] M. J. Savage and R.P. Springer, *Nucl. Phys.* A **644**, 235 (1999).

[19] D. B. Kaplan, M. J. Savage, R. P. Springer and M. B. Wise, *Phys. Lett.* B **449**, 1 (1999).

[20] M. J. Savage, K. A. Scaldeferri, Mark B.Wise, `nucl-th/9811029`.

[21] T. Mehen and I.W. Stewart, `nucl-th/9901064`; `nucl-th/9809095`; *Phys. Lett.* B **445**, 378 (1999).

[22] G. Rupak and N. Shoresh, `nucl-th/9902077`.

[23] J. Gegelia, *Phys. Lett.* B **429**, 227 (1998); `nucl-th/9806028`; `nucl-th/9805008`; `nucl-th/9802038`.

[24] A. K. Rajantie, *Nucl. Phys.* B **480**, 729 (1996).

[25] J.V. Steele and R.J. Furnstahl, *Nucl. Phys.* A **637**, 46 (1998); *Nucl. Phys.* A **645**, 439 (1999).

[26] T.D. Cohen and J.M. Hansen, *Phys. Rev.* C **59**, 13 (1999); `nucl-th/9901065`.

[27] T.-S. Park, K. Kubodera, D.-P. Min, and M. Rho, *Astrophys. Jour.* **507**, 443 (1998); *Phys. Rev.* C **58**, R637 (1998).

[28] X. Kong and F. Ravndal, `nucl-th/9803046`; *Phys. Lett.* B **450**, 320 (1999); `nucl-th/9902064`; `nucl-th/9904026`.

[29] E. Epelbaoum and U.G. Meissner, `nucl-th/9902042`; E. Epelbaoum, W. Glockle, A. Kruger and Ulf-G. Meissner, *Nucl. Phys.* A **645**, 413 (1999); E. Epelbaoum, W. Glockle and Ulf-G. Meissner, *Phys. Lett.* B **439**, 1 (1998); E. Epelbaoum, W. Glockle and Ulf-G. Meissner, *Nucl. Phys.* A **637**, 107 (1998).

[30] T. Mehen, I. W. Stewart and M.B. Wise, `hep-ph/9902370`.

[31] D. R. Phillips, S. R. Beane and M. C. Birse, *J. Phys. A* **32**, 3397 (1999), `hep-th/9810049`.

[32] M. N. Butler and J. W. Chen, `nucl-th/9905059`.

[33] P.F. Bedaque, H.W. Hammer and U. van Kolck, *Phys. Rev. Lett.* **82**, 463 (1999); *Phys. Rev.* C **58**, R641 (1998); P.F. Bedaque and U. van Kolck, *Phys. Lett.* B **428**, 221 (1998).

[34] P. F. Bedaque and H. W. Greisshammer, `nucl-th/9907077`.

[35] T.-S. Park, K. Kubodera, D.-P. Min and M. Rho, Talk presented by M. Rho at the *Nuclear Physics with Effective Field Theory: 1999* workshop, INT, University of Washington, Seattle, February 1999. `nucl-th/9904053`; `nucl-th/9906005`.

[36] J.W. Chen, G. Rupak and M.J. Savage, *Nucl. Phys.* A **653**, 386 (1999).

[37] T.-S. Park, D.-P. Min and M. Rho, *Phys. Rev. Lett.* **74**, 4153 (1995); *Nucl. Phys.* A **646**, 83 (1999).

[38] J.W. Chen, G. Rupak and M.J. Savage, `nucl-th/9905002`.

[39] D. B. Kaplan and J. V. Steele, `nucl-th/9905027`.

[40] S. Fleming, T. Mehen and I. S. Stewart, `nucl-th/9906056`.

[41] D. R. Phillips and T. D. Cohen, `nucl-th/9906091`.

[42] J.W. Chen and M.J. Savage, `nucl-th/9907042`.

[43] M. Rho, `nucl-th/9908015`.

[44] V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen and J.J. de Swart, *Phys. Rev.* C **49**, 2950 (1994), `nucl-th/9406039`.

[45] R. Machleidt, K. Holinde, and C. Elster, *Phys. Rep.* **149**, 1 (1987).

[46] C. W. Wong, *Int. J. Mod. Phys.* **E3**, 821 (1994).

[47] A.J. Buchmann, H. Henning and P.U. Sauer, *Few Body Syst.* **21**, 1996 (149).

[48] J. L. Friar, J. Martorell and D. W. L. Sprung, *Phys. Rev.* A **56**, 4579 (1997).

[49] A. V. Blinov, L. A. Kondratyuk, and I. B. Khriplovich, *Sov. J. Nucl. Phys.* **47**, 382 (1988).

[50] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, *Phys. Rev.* C **51**, 38 (1995), `nucl-th/9408016`.

[51] R. Machleidt, F. Sammarucca, and Y. Song, *Phys. Rev.* C **53**, 1483 (1996), `nucl-th/9510023`.

[52] R. Machleidt, *Advances in Nuclear Physics* **19**, 189 (1989).

[53] J. Adam Jr. , H. Göller and H. Arenhövel, *Phys. Rev.* C **48**, 370 (1993).

[54] J.L. Friar and S. Fallieros, *Phys. Rev.* C **29**, 232 (1984).

[55] J.L. Friar and G.L. Payne, *Phys. Rev.* C **55**, 2764 (1997).